Quantum Effects in the Amplification of Sound in the Presence of a Magnetic Field*

HAROLD N. SPECTOR
Department of Physics, Case Institute of Technology, Cleveland, Ohio
(Received 3 June 1963)

A quantum-mechanical treatment is used to calculate the magnetic-field dependence of sound, which is amplified by interaction with conduction electrons in the presence of crossed dc electric and magnetic fields. It is shown that oscillatory behavior as a function of the strength of the applied magnetic field occurs under conditions of amplification. The oscillations which occur in the amplification have an amplitude equal to the nonoscillatory part of the amplification for nontransverse orientations of sound-wave vector $\mathbf{q}$ and magnetic field $\mathbf{H}$, as long as there are electrons that have a component of drift velocity in the direction of $\mathbf{q}$ which is equal to the sound velocity V_s. The amplification occurs when the drift velocity in the crossed fields, $\mathbf{V}_H$, has a component in the direction of $\mathbf{q}$ which exceeds V_s.

I. INTRODUCTION

A GREAT deal of attention has been paid recently to the study of the amplification of sound waves via their interaction with conduction electrons.[1–8] The previous treatments of this subject are valid only in the semiclassical case where the quantization of the electron orbits can be ignored. However, some of the early evidence for amplification effects in crossed electric and magnetic fields occurred at magnetic field strengths in the quantum limit.[2] It is, therefore, necessary to have a valid quantum-mechanical treatment of the interaction of the sound waves with a gas of conduction electrons.

The procedure used in this paper is that of the self-consistent field method as described by Ehrenreich and Cohen[9] and applied by Zyryanov[10] and Quinn and Rodriguez[11] to calculating the absorption of sound in the quantum limit. In Sec. II, we show how the calculation is carried out and we exhibit explicitly the matrix elements that are of interest. In Sec. III, we apply the results of our calculations to the amplification of sound and in Sec. IV, we give a discussion of our results.

II. DERIVATION OF THE CONDUCTIVITY TENSOR

We shall treat the conduction electrons as a free electron gas of density N_0 confined within a cubic box of side L_0. This gas is in the presence of a magnetic field $\mathbf{B}$ and electric field $\boldsymbol{\mathcal{E}}$ which are at right angles to each other. The sound wave of frequency ω and wave vector $\mathbf{q}$ manifest itself by means of a self-consistent field with scalar and vector potentials $\phi_1(\mathbf{r},t)$, $\mathbf{A}_1(\mathbf{r},t) \propto \exp[i(\mathbf{q}\cdot\mathbf{r}-\omega t)]$. In real materials, the sound wave will also interact with the electron gas by means of a deformation potential. For an extrinsic semiconductor, the electron gas is neutralized by a positive background of the same density while in a semimetal, the electrons are neutralized by an equal number of holes.

The electron current density induced by the self-consistent field is obtained by taking the trace of the current-density operator and the single-particle density matrix. The density-matrix operator ρ must satisfy the equation of motion

$$-i\hbar(\partial\rho/\partial t)=[H,\rho], \tag{2.1}$$

where H is the Hamiltonian of the system. We can separate the Hamiltonian into two parts:

$$H=H_0+H_1, \tag{2.2}$$

where

$$H_0=\frac{1}{2m}(P_x^2+P_z^2)+\frac{1}{2m}(P_y+m\omega_{c}x)^2-e\mathcal{E}_x \tag{2.3}$$

is the Hamiltonian of the electrons in the crossed dc electric and magnetic fields and

$$H_1=-\frac{e}{2c}(\mathbf{v}\cdot\mathbf{A}_1+\mathbf{A}_1\cdot\mathbf{v})+e\phi_1-\mathbf{q}\cdot\frac{\mathbf{C}\cdot\mathbf{u}}{\omega} \tag{2.4}$$

is the part of the Hamiltonian that is of first order in the potentials of the self-consistent field. In (2.3), we have chosen our coordinate system such that the x axis lies along $\boldsymbol{\mathcal{E}}$, the z axis lies along $\mathbf{B}$, and the y axis lies along the third direction of our orthogonal triad. The cyclotron frequency is $\omega_c=eB/mc$ and $\mathbf{v}$ is the velocity operator for the Hamiltonian H_0. In (2.4), the last term arises from the deformation forces, $\mathbf{C}$ being the deformation potential tensor, and $\mathbf{u}$ being the velocity field induced by the passage of the sound wave.

The Hamiltonian H_0 has stationary states charac-

* This work has been supported in part by the U. S. Atomic Energy Commission.

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[9] H. Ehrenreich and M. H. Cohen, Phys. Rev. **115**, 786 (1959).
[10] P. S. Zyryanov, Zh. Eksperim. i Teor. Fiz. **40**, 1065 and 1353 (1961) [translation: Soviet Phys.—JETP **13**, 751 and 953 (1961)]; P. S. Zyryanov and V. P. Kalashnikov, *ibid.* **41**, 1119 (1961) [translation: *ibid.* **14**, 799 (1962)].
[11] J. J. Quinn and S. Rodriguez, Phys. Rev. **128**, 2487 and 2494 (1962).

terized by the wave functions and energy eigenvalues

$$\psi(\mathbf{r})=L_0^{-1}e^{i(k_y y+k_z z)}\phi_n(x-\hbar k_y/m\omega_c+V_H/\omega_c)\,, \tag{2.5a}$$

$$E_{kn}=(n+\tfrac{1}{2})\hbar\omega_c+(\hbar k_z)^2/2m+\hbar k_y V_H-\tfrac{1}{2}mV_H^2\,, \tag{2.5b}$$

where $\phi_n(x)$ is a normalized harmonic oscillator wave function and $V_H=c\mathcal{E}/B$ is the electron drift velocity in the crossed fields. The state of the electron is specified by the quantum numbers k_y, k_z, and n.

We can now solve (2.1) to first order in H_1 by expanding the operator $\rho=\rho_0+\rho_1$, where ρ_0 is the value of ρ in the absence of the sound wave. The equation of motion becomes

$$-i\hbar(\partial\rho_1/\partial t)=[H_0,\rho_1]+[H_1,\rho_0] \tag{2.6}$$

to first order in the self-consistent field. By taking matrix elements of (2.6) in the representaton (2.5), we find

$$\langle k'n'|\rho_1|kn\rangle=\lim_{\delta\to 0}\frac{(f_{kn}-f_{k'n'})\langle k'n'|H_1|kn\rangle}{E_{k'n'}-E_{kn}-\hbar\omega+i\hbar\delta}\,, \tag{2.7}$$

where f_{kn} is the Fermi distribution evaluated at the energy ϵ_{kn}, and ϵ_{kn} is that part of (2.5b) which does not depend upon the electric field.[12]

The current density induced by the passage of the sound wave is

$$\mathbf{j}_1(\mathbf{r},t)=\mathrm{tr}\{\mathbf{j}_{\mathrm{op}}'\rho\}=\tfrac{1}{2}\sum_{kn}\langle kn|e\left(\mathbf{v}-\frac{e}{mc}\mathbf{A}_1\right)\times\delta(\mathbf{r}-\mathbf{x})\varphi+\mathrm{H.c.}|kn\rangle\,, \tag{2.8}$$

where H.c. designates the Hermitian conjugate of the preceding operator. $\mathbf{j}_{\mathrm{op}}'$ is implicitly defined by the second equality in (2.8). The induced charge density $N_1(\mathbf{r},t)$ is obtained from a similar relation.

The relation between the induced current and charge and the self-consistent field potentials are

$$\mathbf{j}'=\frac{\omega_p^2}{4\pi c}\left[\mathbf{W}\cdot\mathbf{A}_1+\mathbf{K}\left(\phi_1+\frac{\mathbf{q}\cdot\mathbf{C}\cdot\mathbf{u}}{e\omega}\right)\right], \tag{2.9a}$$

$$N_1=\frac{\omega_p^2}{4\pi c}\left[-\mathbf{K}^*\cdot\mathbf{A}_1+R\left(\phi_1+\frac{\mathbf{q}\cdot\mathbf{C}\cdot\mathbf{u}}{e\omega}\right)\right], \tag{2.9b}$$

where ω_p is the plasma frequency of the electrons. The symbols $\mathbf{W}$, $\mathbf{K}$, and R stand for

$$\mathbf{W}=1+\frac{m}{N}\sum_{kk'nn'}\frac{[f_{kn}-f_{k'n'}]\langle k'n'|\mathbf{V}|kn\rangle\langle k'n'|\mathbf{V}|kn\rangle^*}{E_{k'n'}-E_{kn}-\hbar\omega+i\hbar\delta}\,, \tag{2.10a}$$

$$\mathbf{K}=\frac{mc}{N}\sum_{kk'nn'}\frac{[f_{kn}-f_{k'n'}]\langle k'n'|\mathbf{V}|kn\rangle\langle kn|e^{i\mathbf{q}\cdot\mathbf{r}}|kn\rangle^*}{E_{k'n'}-E_{kn}-\hbar\omega+i\hbar\delta}\,, \tag{2.10b}$$

and

$$R=\frac{mc^2}{N}\sum_{kk'nn'}\frac{[f_{kn}-f_{k'n'}]|\langle k'n'|e^{i\mathbf{q}\cdot\mathbf{r}}|kn\rangle|^2}{E_{k'n'}-E_{kn}-\hbar\omega+i\hbar\delta}\,. \tag{2.10c}$$

The operator $\mathbf{V}$ which appears in (2.10a–c) is defined by

$$\mathbf{V}=\tfrac{1}{2}[e^{i\mathbf{q}\cdot\mathbf{r}}\mathbf{v}+\mathbf{v}e^{i\mathbf{q}\cdot\mathbf{r}}]\,. \tag{2.11}$$

If we choose $\mathbf{q}$ to lie in the yz plane of our coordinate system, then the matrix elements of $\exp(i\mathbf{q}\cdot\mathbf{r})$ and $\mathbf{V}$ are given by the following equations:

$$\langle k'n'|V_x|kn\rangle=-\delta_{k_z',k_z+q_z}\delta_{k_y',k_y+q_y}i\omega_c\frac{\partial}{\partial q_y}J_{n'n}(q_y)\,, \tag{2.12a}$$

$$\langle k'n'|V_y|kn\rangle=\delta_{k_z',k_z+q_z}\delta_{k_y',k_y+q_y}\left[V_H+(n'-n)\frac{\omega_c}{q_y}\right]J_{n'n}(q_y)\,, \tag{2.12b}$$

$$\langle k'n'|V_z|kn\rangle=\delta_{k_z',k_z+q_z}\delta_{k_y',k_y+q_y}\frac{\hbar}{m}(k_z+\tfrac{1}{2}q_z)J_{nn'}(q_y)\,, \tag{2.12c}$$

$$\langle k'n'|\exp(i\mathbf{q}\cdot\mathbf{r})|kn\rangle=\delta_{k_z',k_z+q_z}\delta_{k_y',k_y+q_y}J_{n'n}(q_y)\,. \tag{2.12d}$$

The symbol $\delta_{k'k}$ is the Kronecker's delta function and $J_{n'n}(q_y)$ is the two-center integral of the harmonic oscillator

[12] The justification for using ϵ_{kn} in the argument of the Fermi distribution instead of E_{kn} is that the chemical potential changes in a way to cancel out the part of E_{kn} which depends on the electric field. See A. H. Kahn and H. P. R. Frederikse, in *Solid State Physics*, edited by F. Seitz and D. Turnbull (Academic Press Inc., New York, 1959), Vol. 9, p. 271.

wave functions defined by

$$J_{n'n}(q)=\int_{-\infty}^{+\infty} dx\, \phi_{n'}(x)\phi_n(x-(\hbar q/m\omega_c)). \tag{2.13}$$

A few of the useful mathematical properties of $J_{n'n}(q)$ are given in the Appendix. Some of them have been used in transforming the matrix elements into the form given in (2.12). These mathematical relations can be used to verify the gauge invariance of (2.9). We find that the following relations hold:

$$\mathbf{W}\cdot\mathbf{q}-(\omega/c)\mathbf{K}=0,\quad \mathbf{K}^*\cdot\mathbf{q}-(\omega/c)R=0. \tag{2.14}$$

When we rewrite (2.9a) in a gauge-invariant form, in terms of the self-consistent electromagnetic field induced by the sound wave we have

$$\mathbf{j}'=\boldsymbol{\sigma}\cdot\left(\boldsymbol{\mathcal{E}}_1-i\mathbf{q}\frac{q\cdot\mathbf{C}\cdot\mathbf{u}}{e\omega}\right), \tag{2.15}$$

where

$$\boldsymbol{\sigma}=(\omega_p^2/4\pi i\omega)\,\mathbf{W} \tag{2.16}$$

is the conductivity tensor.

For the components of the conductivity tensor, using (2.10) and (2.16) together with (2.5b), (2.12), and (2.14), we find

$$\sigma_{xx}=\frac{\omega_p^2}{4\pi i\omega}\left\{1-\frac{m\omega_c^2}{\hbar N_0}\sum_{kn\alpha}\frac{(f_{kn}-f_{k+q,n+\alpha})[\partial J_{n+\alpha n}(q_y)/\partial q_y]^2}{\alpha\omega_c-\omega\mu+(\hbar q_z/m)(k_z+\frac{1}{2}q_z)}\right\}, \tag{2.17a}$$

$$\sigma_{yy}=-\frac{\omega_p^2}{4\pi i\omega}\frac{m}{q_y^2\hbar N_0}\sum_{kn\alpha}\frac{[\omega-(\hbar q_z/m)(k_z+\frac{1}{2}q_z)]^2(f_{kn}-f_{k+q,n+\alpha})J_{n+\alpha,n}^2(q_y)}{\alpha\omega_c-\omega\mu+(\hbar q_z/m)(k_z+\frac{1}{2}q_z)}, \tag{2.17b}$$

$$\sigma_{zz}=\frac{\omega_p^2}{4\pi i\omega}\frac{m}{q_z^2\hbar N_0}\sum_{kn\alpha}\frac{[\omega\mu-\alpha\omega_c]^2(f_{kn}-f_{k+q,n+\alpha})J_{n+\alpha,n}^2(q_y)}{\alpha\omega_c-\omega\mu+(\hbar q_z/m)(k_z+\frac{1}{2}q_z)}, \tag{2.17c}$$

$$\sigma_{xy}=-\sigma_{yx}=\frac{\omega_p^2}{8\pi\omega q_y}\frac{m\omega_c}{\hbar N_0}\sum_{kn\alpha}\frac{[\omega-(\hbar q_z/m)(k_z+\frac{1}{2}q_z)](f_{kn}-f_{k+q,n+\alpha})(\partial J_{n+\alpha,n}^2/\partial q_y)(q_y)}{\alpha\omega_c-\omega\mu+(\hbar q_z/m)(k_z+\frac{1}{2}q_z)}, \tag{2.17d}$$

$$\sigma_{yz}=\sigma_{zy}=\frac{\omega_p^2}{4\pi i\omega q_z}\frac{m}{\hbar N_0}\sum_{kn\alpha}\frac{[\omega-(\hbar q_z/m)(k_z+\frac{1}{2}q_z)](f_{kn}-f_{k+q,n+\alpha})(\hbar/m)(k_z+\frac{1}{2}q_z)J_{n+\alpha,n}^2(q_y)}{\alpha\omega_c-\omega\mu+(\hbar q_z/m)(k_z+\frac{1}{2}q_z)}, \tag{2.17e}$$

$$\sigma_{xz}=-\sigma_{zx}=\frac{\omega_p^2}{8\pi\omega}\frac{m\omega_c}{\hbar N_0}\sum_{kn\alpha}(f_{kn}-f_{k+q,n+\alpha})\frac{\hbar}{m}(k_z+\tfrac{1}{2}q_z)\frac{\partial}{\partial q_y}J_{n+\alpha,n}^2(q_y), \tag{2.17f}$$

where $\mu=1-\hat{q}\cdot\mathbf{V}_H/V_s$ and $\hat{q}$ is a unit vector in the direction of propagation.

III. THE ABSORPTION COEFFICIENT

The quantity that is of interest experimentally in studying the interaction between the sound wave and the conduction electrons is the absorption coefficient α. This coefficient gives the exponential change of sound intensity with distance. The absorption coefficient is the average power density transferred between the sound wave and the electrons per unit energy flux, or

$$\alpha=Q/\tfrac{1}{2}\rho|\mathbf{u}|^2V_s, \tag{3.1}$$

where ρ is the density of the material.

In a semimetal, the net power transferred per unit volume is

$$Q=\tfrac{1}{2}\,\mathrm{Re}\left\{(\mathbf{j}_e+\mathbf{j}_n)^*\boldsymbol{\mathcal{E}}_1+\mathbf{j}_e^*\cdot\mathbf{q}\frac{\mathbf{C}_e\cdot\mathbf{u}}{ei\omega}-\frac{\mathbf{j}_h^*\cdot\mathbf{q}\cdot\mathbf{C}_n\cdot\mathbf{u}}{ei\omega}\right\}, \tag{3.2}$$

where the subscript e denotes quantities associated with the electrons and h denotes those associated with the holes. The self-consistent field arising from the electron and hole currents can be obtained from Maxwell's equations[7,13] and is

$$\mathbf{j}_e+\mathbf{j}_h=(4\pi/i\omega)\{(c/V_s)^2\mathbf{I}-[(c/V_s)^2+1]\hat{q}\hat{q}\}\boldsymbol{\mathcal{E}}_1, \tag{3.3}$$

where $\mathbf{I}$ is the unit matrix.

Using (3.2) and (3.3) together with (2.15), we can

[13] M. J. Harrison, Phys. Rev. **119**, 1260 (1960).

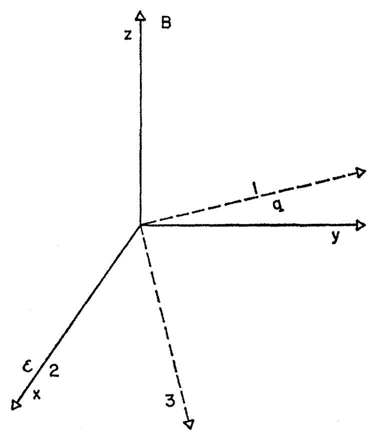

FIG. 1. The relationship between the coordinate systems (x,y,z) and $(1,2,3)$ is shown. The system (x,y,z) is fixed by the directions of the fields $\boldsymbol{\mathcal{E}}$ and $\mathbf{B}$. The system $(1,2,3)$ is fixed by the direction of propagation $\mathbf{q}$.

calculate α for a semimetal. For semimetals, the forces arising from the deformation potential dominate the electrostatic forces for sound frequencies greater than 1 Mc/sec. It is only when the deformation potential forces do dominate, that we have appreciable interaction between the sound wave and the electrons in a material with low carrier density.[13] In the region where the deformation forces dominate the interaction, we find

$$\alpha_j=\frac{2N_0m}{\rho V_s}\left(\frac{C_{1j}}{mV_s^2}\right)^2\left(\frac{\omega}{\omega_p}\right)^2\mathrm{Re}\sigma_{11}, \tag{3.4}$$

where we have chosen the 1 direction of our coordinate system to lie along $\mathbf{q}$ and the subscript j denotes the direction of polarization of the wave. In deriving (3.4), we have assumed, for the sake of simplicity, that the masses and deformation potentials of the holes and electrons are equal.

To calculate the absorption coefficient, we need only know the σ_{11} component of the conductivity tensor. This component can be calculated by performing a transformation from the coordinate system of (2.17) to our new coordinate system as shown in Fig. 1. We find that

$$\sigma_{11}=\frac{\omega_p^2}{4\pi i}\frac{m\omega}{q^2\hbar N_0}\sum_{k_yk_zn\alpha}\frac{(f_{kn}-f_{k+q,n+\alpha})J_{n+\alpha,n}^2(q_y)}{\alpha\omega_c-\omega\mu+(\hbar q_z/m)(k_z+\frac{1}{2}q_z)}. \tag{3.5}$$

The real part of σ_{11} can be evaluated by using the relation[14]

$$\lim_{\delta\to0^+}\frac{1}{z+i\delta}=P\frac{1}{z}+i\pi\delta(z), \tag{3.6}$$

where $P(1/z)$ indicates that in any integration, the principal part of the integral is to be taken.

Using (3.4)–(3.6), we find the absorption coefficient is

$$\alpha_j=\frac{N_0m}{2\rho V_s}\left(\frac{C_{1j}}{mV_s^2}\right)^2\left(\frac{\omega}{\omega_p}\right)^2\frac{\omega_p^2\omega}{q^2}\frac{m}{\hbar N_0}\sum_{k_yk_zn\alpha}(f_{kn}-f_{k+q,n+\alpha})$$
$$\times J_{n+\alpha,n}^2(q_y)\delta\left(\alpha\omega_c-\omega\mu+\frac{\hbar q_z}{m}(k_z+\tfrac{1}{2}q_z)\right). \tag{3.7}$$

We shall now evaluate (3.7) explicitly for two different cases. In case A, we shall take the direction of propagation to be transverse to the magnetic field while in case B, $\mathbf{q}$ will have a component in the direction of $\mathbf{B}$.

A. Propagation in a Transverse Magnetic Field $\mathbf{q}\perp\mathbf{B}$.

In this situation we find that the absorption coefficient can be written as

$$\alpha_j=\frac{N_0m}{2\rho V_s}\left(\frac{C_{1j}}{mV_s^2}\right)^2\frac{\omega^4\mu}{q^2}\frac{m}{N_0}\sum_{k_yk_zn\alpha}\frac{\partial f_{kn}}{\partial\varphi}$$
$$\times J_{n+\alpha,n}^2(q_y)\delta(\alpha\omega_c-\omega\mu), \tag{3.8}$$

where φ is the Fermi energy. We have expanded $f_{kn}-f_{k+q,n+\alpha}$ in the following fashion:

$$f_{kn}-f_{k+q,n+\alpha}=(\partial f_{kn}/\partial\varphi)(\epsilon_{kn}-\epsilon_{k+q,n+\alpha})$$
$$=\hbar\alpha\omega_c(\partial f_{kn}/\partial\varphi). \tag{3.9}$$

From (3.8), we can see that we have resonant absorption or amplification of sound, whenever $\omega\mu=\alpha\omega_c$. When $\mu>0$, the sound is attenuated and when $\mu<0$, the sound is amplified.

It is interesting to note the form (3.8) takes in the semiclassical limit. In this case, we deal with high quantum numbers and we can replace the summation over n by an integration over θ from 0 to $\frac{1}{2}\pi$, where $n=n_0\sin^2\theta$ and $n_0=\varphi/\hbar\omega_c-\frac{1}{2}$. We can then replace $J_{n,n+\alpha}(q_y)$ by its asymptotic form for large[15] n:

$$\lim_{n\to\infty}J_{n,n+\alpha}(q_y)=J_\alpha([2n\hbar q_y^2/m\omega_c]^{1/2}). \tag{3.10}$$

Here $J_\alpha(x)$ is the Bessel function of order α and argument x. In this limit, (3.8) takes the form

$$\alpha_j=\frac{3N_0m}{2\rho V_s}\left(\frac{C_{1j}}{mV_s^2}\right)^2\frac{\omega\mu}{(qV_F)^2}\sum_\alpha g_\alpha(x)\delta(\alpha\omega_c-\omega\mu), \tag{3.11}$$

where V_F is the Fermi velocity, $x=q_yV_F/\omega_c$, and

$$g_\alpha(x)=\int_0^{\pi/2}d\theta\,\sin\theta J_\alpha^2(x\sin\theta). \tag{3.12}$$

This answer agrees with the results obtained by using

[14] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Book Company, Inc., New York, 1953), Vol. 1, p. 473.

[15] A. Erdelye, W. Magnus, F. Oberhettinger, and F. G. Tricomi, in *Higher Transcendental Functions*, edited by A. Erdelyi (McGraw-Hill Book Company, Inc., New York, 1953), Vol. 2, p. 199.

the Boltzmann equation in the limit of infinite relaxation time.[16]

In real semimetals, there is a finite relaxation time which limits the peak values of the attenuation and amplification. We can take account of collisions with impurities by adding a term $i\hbar(\rho-\rho_s)/\tau$ to the left side of the equation of motion of the density-matrix operator (2.6). The density matrix relaxes, in the presence of the sound wave, to an equilibrium distribution which is centered about the impurity velocity and which depends on the local value of the Fermi energy. It is possible to expand ρ_s about the Fermi equilibrium distribution ρ_0. This procedure gives two correction terms. The first correction is equivalent to adding a fictitious electric field of magnitude $m\mathbf{u}/e\tau$ to the true electric field. The second correction gives rise to a diffusion current which adds a term $-\mathbf{R}N_1eV_s$ to (2.15), where $\mathbf{R}$ is the diffusion vector.[11,16] There remains in the equation of motion of ρ a term $i\hbar(\rho-\rho_0)/\tau$, which can be formally taken account of in the expressions for $\boldsymbol{\sigma}$ and $\mathbf{R}$ by replacing ω by $\omega+i/\tau$. When all the corrections have been accounted for, we end up with an expression for the absorption coefficient like (3.4), where σ_{11} has been replaced by an effective conductivity σ_{11}';

$$\sigma_{11}'=\sigma_{11}/(1-R)_1\,, \tag{3.13}$$

and

$$\sigma_{11}=\frac{3\sigma_0(1-i\omega\tau)}{(ql)^2}\left[1-\frac{2}{3}\frac{\varphi}{N_0}(1-i\omega\mu\tau)\times\sum_{k_yk_zn\alpha}\frac{(\partial f_{kn}/\partial\varphi)J_{n+\alpha,n}^2(q)}{1+i(\alpha\omega_c-\omega\mu)\tau}\right], \tag{3.14a}$$

$$R_1=\frac{1}{i\omega\tau}\left[1-\frac{2}{3}\frac{\varphi}{N_0}\sum_{k_yk_zn\alpha}\frac{(\partial f_{kn}/\partial\varphi)J_{n+\alpha,n}^2(q)}{1+i(\alpha\omega_c-\omega\mu)\tau}\right]. \tag{3.14b}$$

In (3.14a–b), σ_0 is the dc conductivity and l is the electron mean-free path, $l=V_F\tau$.

Using (3.10) in (3.14a–b), we can find the limiting expressions for σ_{11} and R_1 in the semiclassical limit:

$$\sigma_{11}=\frac{3\sigma_0(1-i\omega\tau)}{(ql)^2}\times\left[1-(1-i\omega\mu\tau)\sum_\alpha\frac{g_\alpha(x)}{1+i(\alpha\omega_c-\omega\mu)\tau}\right], \tag{3.15a}$$

$$R_1=\frac{1}{i\omega\tau}\left[1-(1-i\omega\tau)\sum_\alpha\frac{g_\alpha(x)}{1+i(\alpha\omega_c-\omega\mu)\tau}\right]. \tag{3.15b}$$

In the region of cyclotron resonance, i.e., $\omega=\alpha\omega_c$, $x=(\omega/\omega_c)(V_F/V_s)\gg1$, and we obtain the following for $\mathrm{Re}\sigma_{11}'$:

$$\mathrm{Re}\sigma_{11}'=\frac{3}{8}\frac{\omega_p^2}{\omega}\left(\frac{V_s}{V_F}\right)^3\mu\ \mathrm{Re}\coth\pi\frac{(1-i\omega\mu\tau)}{\omega_c\tau}. \tag{3.16}$$

In the region of geometric resonance, $x\sim1$, and we can usually also satisfy the condition $|\omega_c\tau/(1-i\omega\mu\tau)|^2\gg1$, so that we need keep only the $\alpha=0$ term in the summation over α. Therefore, in this region we have

$$\mathrm{Re}\sigma_{11}'=\frac{3\omega_p^2}{4\pi}\left(\frac{V_s}{V_F}\right)^2\frac{\omega\mu\tau g_0(x)[1-g_0(x)]}{[1-g_0(x)]^2+(\omega\mu\tau)^2}. \tag{3.17}$$

We have a maximum value for the absorption coefficient when

$$\mu=\pm(1-g_0(x))/\omega\tau\,, \tag{3.18}$$

and this maximum value is

$$\alpha_{j\,\max}=\pm\frac{3N_0m}{\rho V_s}\left(\frac{C_{1j}}{mV_s}\right)^2\left(\frac{V_s}{V_F}\right)^2\omega g_0(x). \tag{3.19}$$

The results (3.16)–(3.19) are the same as obtained previously using a Boltzmann equation approach[8] and are shown just for comparison.

In going to the region where quantum effects are important, we shall consider the case where $|\omega_c\tau/(1-i\omega\mu\tau)|^2\gg1$ and $\hbar q^2/2m\omega_c\ll1$. We can use the following limiting form for $J_{nn}^2(q)$[17]:

$$J_{nn}^2(q)=1-(n+\tfrac{1}{2})(\hbar q^2/m\omega_c)\,, \tag{3.20}$$

using the expansion of the Laguerre polynomials. We now have

$$\sigma_{11}=\frac{\sigma_0(1-i\omega\tau)}{2(ql)^2}\frac{\varphi}{N_0}\frac{\hbar q^2}{m\omega_c}\sum_{k_yk_zn}\frac{\partial f_{kn}}{\partial\varphi}(n+\tfrac{1}{2})\,, \tag{3.21a}$$

$$R_1=\frac{1}{1-i\omega\mu\tau}\left[\frac{V_H}{V_s}+\frac{2\varphi}{3N_0}\frac{(1-i\omega\tau)}{i\omega\tau}\frac{\hbar q^2}{m\omega_c}\sum_{k_yk_zn}\frac{\partial f_{kn}}{\partial\varphi}(n+\tfrac{1}{2})\right]. \tag{3.21b}$$

The summation over the quantum numbers k_yk_zn can be performed using a method which has been described in detail by Wilson[18]:

$$\sum_{k_yk_zn}\frac{\partial f_{kn}}{\partial\varphi}(n+\tfrac{1}{2})=\frac{N_0}{\hbar\omega_c}(1+F)\,, \tag{3.22}$$

where

$$F=3\sqrt{2}\pi^2\frac{kT}{\hbar\omega_c}\left(\frac{\hbar\omega_c}{\varphi}\right)^{1/2}\frac{\cos(2\pi r\varphi/\hbar\omega_c-\frac{1}{4}\pi)}{\sinh(2\pi^2rkT/\hbar\omega_c)}. \tag{3.23}$$

F is an oscillatory function of magnetic field, whose origin is the same as that of the de Haas-van Alphen oscillations of the magnetic susceptibility of semimetals.

[16] M. H. Cohen, M. J. Harrison, and W. A. Harrison, Phys. Rev. **117**, 937 (1960).

[17] Ref. 15, p. 188.

[18] A. H. Wilson, *The Theory of Metals* (Cambridge University Press, New York, 1958), 2nd ed., pp. 160–168.

In this limit the absorption coefficient is

$$\alpha_j = \frac{2N_0 m}{\rho V_s}\left(\frac{C_{1j}}{mV_s^2}\right)^2 \frac{\omega^2\mu\tau(\omega/\omega_c)^2(1+F)}{(\omega\mu\tau)^2+\frac{1}{9}x^4(1+F)^2}. \quad (3.24)$$

Therefore, for $\mathbf{q}\perp H$, we have a quantum oscillation of small amplitude superimposed on the ordinary semiclassical absorption coefficient. We have a maximum in the absorption coefficient at fields such that

$$\mu = \pm\frac{1}{3}\frac{x^2}{\omega\tau}(1+F) \quad (3.25)$$

at these fields, the absorption coefficient has the value

$$\alpha_{j\,\max} = \pm\frac{3N_0 m}{\rho V_s}\left(\frac{C_{1j}}{mV_s^2}\right)^2\left(\frac{V_s}{V_F}\right)^2\omega, \quad (3.26)$$

which is the same as the maximum value obtained in the semiclassical case in the high-field limit. Thus, for propagation transverse to the magnetic field, there is no quantum correction to the absorption coefficient in the limit of infinite relaxation time. In all cases, we have amplification when $\mu<0$.

B. Direction of Propagation Not Transverse to Magnetic Field $\mathbf{q}\cdot\mathbf{B}\neq 0$

In the situation when there is a component of $\mathbf{q}$ along $\mathbf{B}$, we must go back to (3.7). However, we will assume that the condition $\omega_c \gg \omega$ is satisfied so that we need keep only the $\alpha=0$ term in the summation over α. This condition will be satisfied except in the region where cyclotron resonance would occur. As in (3.8), we can expand $f_{kn}-f_{k+q,n}$ and we obtain

$$\alpha_j = \frac{N_0 m}{2\rho V_s}\left(\frac{C_{1j}}{mV_s^2}\right)^2 \frac{\omega^4\mu}{q^2 q_z}\frac{m^2}{N_0\hbar}\frac{\partial}{\partial\varphi}\sum_{k_y k_z n} f_0(\epsilon_{kn}-\varphi) \times J_{nn}^2(q_y)\delta(k_z - m\omega\mu/\hbar q_z + \tfrac{1}{2}q_z). \quad (3.27)$$

We can readily perform the summations over k_y and k_z and we find

$$\alpha_j = \frac{3N_0 m}{4\rho V_s}\left(\frac{C_{1j}}{mV_s^2}\right)^2 \frac{\omega^4\mu}{(qV_F)^2}\frac{\hbar\omega_c}{q_z V_F}\frac{\partial}{\partial\varphi}\sum_n J_{nn}^2(q_y) \times f_0((n+\tfrac{1}{2})\hbar\omega_c - \xi), \quad (3.28)$$

where

$$\xi = \varphi - (\hbar^2/2m)(m\omega\mu/\hbar q_z - \tfrac{1}{2}q_z)^2, \quad (3.29)$$

and we have used the fact that

$$N_0 = (1/3\pi^2)(mV_F/\hbar)^3. \quad (3.30)$$

We can see from (3.28) that we have amplification when $\mu<0$ and attenuation when $\mu>0$. We can perform the summation over n by using Poisson's summation formula.[19] The Poisson's summation formula tells us that

$$\sum_{n=0}^{\infty}\phi(n+\tfrac{1}{2}) = \sum_{r=-\infty}^{+\infty}(-1)^r\int_0^\infty dx\, e^{2\pi i r x}\phi(x). \quad (3.31)$$

In the limit of high magnetic fields where $\hbar q_y^2/2m\omega_c \ll 1$, $J_{nn}^2(q_y)\approx 1$, and we have the following summation over n:

$$\sum_{n=0}^{\infty} f_0((n+\tfrac{1}{2})\hbar\omega_c - \xi) = \sum_{r=-\infty}^{+\infty}(-1)^r\int_0^\infty \frac{dx\, e^{2\pi i r x}}{\exp[x\hbar\omega_c - \xi]/kT+1}. \quad (3.32)$$

The integration over x can be performed and we find that

$$\sum_{n=0}^{\infty} f_0((n+\tfrac{1}{2})\hbar\omega_c - \xi) = \frac{\xi}{\hbar\omega_c} + \frac{2\pi kT}{\hbar\omega_c}\sum_{r=1}^{\infty}\frac{(-1)^r\sin(2\pi r\xi/\hbar\omega_c)}{\sinh(2\pi^2 rkT/\hbar\omega_c)}. \quad (3.33)$$

Using (3.33) in (3.28), we find for α

$$\alpha_j = \frac{3N_0 m}{4\rho V_s}\left(\frac{C_{1j}}{mV_s}\right)^2 \frac{\omega^4\mu}{(qV_F)^2 q_z V_F} \times\left\{1+\frac{4\pi^2 kT}{\hbar\omega_c}\sum_{r=1}^{\infty}\frac{(-1)^r r\cos(2\pi r\xi/\hbar\omega_c)}{\sinh(2\pi^2 rkT/\hbar\omega_c)}\right\}. \quad (3.34)$$

We see that we can get quantum resonances in the amplification of sound, whenever $\xi/\hbar\omega_c$ takes on an integer value as long as $kT \ll \hbar\omega_c$. The parameter ξ is related to the cross-sectional area of the orbit on the Fermi surface that is in resonance with the sound wave. This orbit is the one that has a drift velocity which satisfies the relation

$$\hat{q}\cdot\mathbf{V}_d = V_s, \quad \mathbf{V}_d = \mathbf{V}_H + V_z\hat{B}. \quad (3.35)$$

As we vary the strength of the magnetic field, we take various Landau levels through this cross-sectional area. The relation between ξ and the cross-sectional area of this orbit s is

$$\xi = s/2\pi m. \quad (3.36)$$

These quantum oscillations can only be observed if the energy spacing between Landau levels is not smeared out by the thermal broadening of these levels.

When we go to the limit of high quantum numbers we can again replace $J_{nn}(q_y)$ by its asymptotic form (3.10)

[19] R. Courant and D. Hilbert, *Methods of Mathematical Physics* (Interscience Publishers, Inc., New York, 1953), Vol. 1, p. 76.

and we have

$$\frac{\partial}{\partial\varphi}\sum_{n=0} J_0{}^2\left(\left[\frac{2n\hbar q_y{}^2}{m\omega_c}\right]^{1/2}\right)f_0((n+\tfrac{1}{2})\hbar\omega_c-\xi)$$
$$=\sum_{r=-\infty}^{+\infty}(-1)^r\frac{\partial}{\partial\xi}\int_0^\infty dx\, e^{2\pi irx}J_0{}^2\left(\left[\frac{2x\hbar q_y{}^2}{m\omega_c}\right]^{1/2}\right)\times f_0(x\hbar\omega_c-\xi)\,, \quad (3.37)$$

where we have again used Poisson's summation formula. Since $\partial f_0/\partial\xi$ acts at low temperatures like the Dirac delta function, we can take the more slowly varying Bessel function out of the integral and replace it by its value at $x=\xi/\hbar\omega_c$. We can then perform the remainder of the integration over x as in (3.33). In this limit we obtain the following result

$$\alpha_j=\frac{3N_0m}{4\rho V_s}\left(\frac{C_{1j}}{mV_s{}^2}\right)^2\frac{\omega^4\mu}{(qV_F)^2q_zV_F}J_0{}^2\left(\frac{q_y}{\omega_c}\left(\frac{\xi}{m}\right)^{1/2}\right)\times\left[1+\frac{4\pi^2kT}{\hbar\omega_c}\sum_{r=1}^{\infty}\frac{(-1)^r r\cos(2\pi r\xi/\hbar\omega_c)}{\sinh(2\pi^2kT/\hbar\omega_c)}\right]. \quad (3.38)$$

Here we have two different oscillatory terms. The oscillations arising from the Bessel function are the geometric resonances for the orbits in phase with the sound wave. These geometric resonances occur classically and arise from the matching of orbit diameters with integral numbers of sound wavelengths. Superimposed on the geometric resonances, in the case where the spacing of Landau levels $\hbar\omega_c$ is greater than the thermal broadening kT, are the de Haas-Schubnikov oscillations. The periods of the geometric resonances are related to the dimension of the orbit which satisfies the resonance condition (3.35), while the periods of the de Haas-Schubnikov oscillations are related to the cross-sectional areas of these same orbits.

IV. DISCUSSION

In our calculations, we have shown that quantum oscillations in the sound intensity can occur under conditions of amplification. This happens when the drift velocity imparted to the conduction electrons in the crossed electric and magnetic fields is greater than the sound velocity. Under these conditions, the conduction electrons can radiate phonons in analogy with the Čerenkov radiation of light. Moreover, in the presence of the crossed fields, we have found that there is a resonant transfer of energy between the sound wave and the conduction electrons. This resonant transfer of energy occurs when there are electrons that are in phase with the sound wave, i.e., when there are electrons that have a component of drift velocity in the direction of propagation which is equal to the sound velocity V_s. In the absence of the dc electric field, this can only occur when $\mathbf{q}$ has a component in the direction of the magnetic field so that the condition

$$V_z\hat{q}\cdot\hat{B}=V_s \quad (4.1)$$

is satisfied. In the presence of the electric field, the resonant transfer of energy can occur for arbitrary orientation of $\mathbf{q}$ and $\mathbf{B}$.

We find several different kinds of oscillations arising in the amplification of sound. When the direction of propagation is not transverse to the dc magnetic field, we can have geometric resonances superimposed upon de Haas-Schubnikov quantum oscillations. The geometric resonances arise from the matching of diameters of the cyclotron orbit with the sound wavelength. The quantum oscillations arise when the cross section of the orbit is an integral multiple of ehB/c. The quantum oscillations have an amplitude which is equal in magnitude to the semiclassical amplification. Since the diameters and the cross-sectional areas are those of the orbits which are in resonance with the sound wave, we can obtain information about these quantities for non-extremal orbits on the Fermi surface. Thus, by varying the angle between $\mathbf{q}$ and $\mathbf{B}$, we can bring different orbits into resonance with the sound wave and map out the Fermi surface. The fact that this mapping can be done under conditions of amplification makes the possibility of their study more favorable than it is at present. This resonant transfer of energy between the sound wave and the conduction electrons can only occur when $\omega\tau\gg1$. At frequencies high enough to satisfy this condition, the attenuation is usually too large to measure anything conveniently. Under conditions of amplification, however, this problem would not arise since we would only get large amplification factors where the resonances occur.

When the direction of propagation is transverse to the magnetic field, we find quantum oscillations of small amplitude [of order $(\hbar\omega_c/\varphi)^{1/2}$] superimposed upon the semiclassical amplification coefficient. However, in the limit of infinite relaxation time the maxima of the amplification become independent of any quantum oscillations as long as $qV_F/\omega_c\ll1$. This is evidence that the quantum oscillations in the amplification for $\mathbf{q}\perp\mathbf{B}$ arise mainly because of the scattering of the electrons by impurities and other scattering mechanisms.

The effects discussed in this paper will only occur in semimetals and degenerate semiconductors. In metals, the conductivity is too high to obtain the dc electric fields necessary to cause V_H to exceed V_s. The conditions needed for observing these quantum effects would require very pure materials at low temperatures. These conditions might best be satisfied in semimetals like Bi and Sb, where there is already evidence of quantum oscillations.[20,21]

[20] A. P. Korolyuk and T. A. Pruschak, Zh. Eksperim. i Teor. Fiz. **41**, 1689 (1961) [translation: Soviet Phys.—JETP **14**, 1201 (1962)].

[21] J. B. Ketterson, Phys. Rev. **129**, 18 (1963).

APPENDIX

The purpose of this Appendix is to give a few mathematical properties of the matrix elements $J_{n'n}(q)$. This quantity is defined by (2.13) and can be put in the form

$$J_{n'n}(q)=\left(\frac{n!}{n'!}\right)^{1/2}\left(\frac{\hbar q^2}{2m\omega_c}\right)^{\frac{1}{2}(n'-n)} \times\exp-\frac{\hbar q^2}{4m\omega_c}L_n^{n'-n}\left(\frac{\hbar q^2}{2m\omega_c}\right), \quad (A1)$$

by using the properties of the harmonic oscillator functions $\phi(x)$. The formula (A1) is only valid for $n'\geq n$. $L_n{}^{\alpha}(x)$ is an associated Laguerre polynomial. An expression similar to (A1) can be found when $n'<n$ by using the relations

$$J_{n'n}(-q)=J_{nn'}(q)=(-1)^{n'-n}J_{n'n}(q). \quad (A2)$$

Using the properties of the Laguerre polynomials, we are able to derive the relations

$$\frac{\partial J_{n'n}}{\partial q}=\left(\frac{\hbar}{2m\omega_c}\right)^{1/2}[(n+1)^{1/2}J_{n',n+1}-n^{1/2}J_{n',n-1}], \quad (A3)$$

$$\left(n'-n-\frac{\hbar q^2}{2m\omega_c}\right)J_{n'n}(q) =\left(\frac{\hbar q^2}{2m\omega_c}\right)^{1/2}[(n+1)^{1/2}J_{n',n+1}+n^{1/2}J_{n',n-1}], \quad (A4)$$

which we have used to simplify the matrix elements (2.12). We can also obtain the useful sum rules

$$\sum_{n'=0}^{\infty} J_{n'n}{}^2(q)=1, \quad (A5)$$

$$\sum_{n'=0}^{\infty} (n'-n)J_{n'n}{}^2(q)=\frac{\hbar q^2}{2m\omega_c}. \quad (A6)$$

PHYSICAL REVIEW VOLUME 132, NUMBER 2 15 OCTOBER 1963

Effects of an Electric Field on Molecular Excitons

ARTHUR BIERMAN
Space Science Laboratory, Lockheed-California Company, Burbank, California
(Received 4 June 1963)

The influence of an electric field on the second moment $\Delta(t)$ of an exciton wave packet is calculated. The following formula is derived: $\Delta(t)=(2\beta/\hbar^2)[\beta+2\lambda_1(B_1+C_1)]t^2$, where λ_1 is the (uniform) strength of the field along the linear chain molecule and the term B_1+C_1 depends on the parameters of the system. The gradient of the electric field does not contribute to $\Delta(t)$. It is also shown that both the exciton electric dipole moment and B_1+C_1 vanish unless either some states of the units making up the chain (monomers) are parity mixtures (as in molecules), or the coupling potential between monomers is not symmetric with respect to the parity operators of pairs of adjacent monomers. It must also be required that the monomers have zero static dipole moment for the state corresponding to the exciton.

INTRODUCTION

IN a previous paper, herein referred to as (A), the author[1] has derived an expression for the acceleration of an exciton wave packet due to an external electric field. The acceleration was shown to be proportional to the gradient of the electric field, the proportionality constant being, therefore, interpretable as the exciton electric dipole moment. In the present paper, we extend the analysis by (a) investigating the effect of the electric field on the rate of spreading of the wave packet, i.e., on the *second* moment of the exciton distribution function, and (b) carrying out a brief evaluation of some of the derived physical constants of the theory, including the exciton dipole moment. All assumptions of the first paper are preserved.

[1] A. Bierman, Phys. Rev. **130**, 2266 (1963).

THE SECOND MOMENT OF THE EXCITON WAVE PACKET

We define the second moment by

$$\Delta(t)=\sum_k \xi_k'^*\xi_k'[k-\langle x\rangle]^2=\sum_k k^2\xi_k'^*\xi_k'-\langle x\rangle^2. \quad (1)$$

The average position $\langle x\rangle$ can be trivially calculated from Eqs. (17) and (71) of (A). Since the wave packet moves with constant acceleration a, and the initial velocity v_0 is given by

$$v_0=\frac{1}{\hbar i}\sum_{k,l} H_{kl}(k-l)\xi_k^*(0)\xi_l(0) =\frac{1}{\hbar i}\sum_{k,l}(k-l)H_{kl}\delta_{k0}\delta_{l0}=0, \quad (2)$$

we find

$$\langle x\rangle=\tfrac{1}{2}at^2.$$